Common Fixed Point Theorms in 2 Non-Archimedean Menger PM-Space Using (EA) Property

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Abstract – We show some common fixed point theorems in 2 non-Archimedean Menger PM-space by using the property (E.A) and non-compatibility for a pair and quadruplet of self-mappings. A few connected results and helpful examples are also discussed.

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Index Terms – 2 non-Archimedean Menger PM-space, Non-compatible maps, Fixed points and property (E.A).

1. INTRODUCTION

In 1986, Jungck [1] introduced the concept of compatible mappings. A number of fixed point theorems have been obtained by different authors in metric spaces, probabilistic metric spaces and fuzzy metric spaces using the conception of compatibility of maps or by using its generalized or weaker forms. Pant [8] initiated the work using the concept of non-compatible maps in metric spaces. Recently, Aamri and Moutawakil [5] introduced the property (E.A) and thus generalized the concept of non-compatibility.

In this paper, we show common fixed point theorems for R-weakly commuting maps in 2 non-Archimedean Menger PM-space by using the concept of non-compatibility or its generalized notion that is the property (E.A). Our results unify, extend and generalize the results of G. Jungck [1, 2] , R. P. Pant [7, 8, 9, 10], H. K. Pathak et al. [3] and S. Padaliya et al. [13].

2. PRELIMINARIES

Renu Chugh and Sumitra [11] introduced the following definitions:

Definition 2.1:

Let X be any non-empty set and D be the set of all left continuous distribution functions. An ordered pair (X , F) is said to be 2 non-Archimedean probabilistic metric space (briefly 2 N.A. PM-space) if F is a mapping from $X \times X \times X$ into D satisfying the following conditions, where the value of F at $(x, y, z) \in X \times X \times X$ represented by $F_{x,y,z}$ or $F(x, y, z)$ for each $x, y, z \in X$ such that

(i) F(x, y, z; t) = 1 for all t > 0 if and only if at least two of the three points are equal.

(ii) F(x, y, z) = F(x, z, y) = F(z, x, y)

(iii) F(x, y, z; 0) = 0

(iv) If

$$F(x, y, s; t_1) = F(x, s, z; t_2) = F(s, y, z; t_3) = 1,$$
$$\text{then } F(x, y, z; \max\{t_1, t_2\, t_3\}) = 1$$

Definition 2.2:

A t-norm is a function $\Delta : [0,1] \times [0,1] \times [0,1] \to [0,1]$ which is associative, commutative, non decreasing in each coordinate and $\Delta(a,1,1) = a$ for each $a \in [0,1]$.

Definition 2.3:

A 2 N.A. Menger PM-space is an ordered triplet (X, F, Δ), where Δ is a t-norm and (X , F) is a 2 N.A. PM-space satisfying the following condition

$$F(x, y, z; \max\{t_1, t_2, t_3\}) \geq (\Delta(F(x, y, s; t_1), F(x, s, z; t_2), F(s, y, z; t_3)))$$
$$\text{for each } x, y, z \in X, t_1, t_2, t_3 \geq 0$$

Definition 2.4:

Let (X, F, Δ) be 2 N.A. Menger PM-space and Δ a continuous t-norm, then (X, F, Δ) is Hausdorff in the topology induced by the family of neighborhoods, $U_x(\varepsilon, \lambda, a_1, a_2, ..., a_n); x, a_i \in X, \varepsilon > 0, i = 1, 2, ..., n \in Z^+$, where Z^+ is the set of all positive integers and

$$U_x(\varepsilon,\lambda,a_1,a_2,.....,a_n) =$$
$$\{y \in X; F(x,y,a_i;\varepsilon) > 1-\lambda, 1 \le i \le n\}$$
$$= \bigcap_{i=1}^{n} \{y \in X; F(x,y,a_i;\varepsilon) > 1-\lambda, 1 \le i \le n\}.$$

Definition 2.5:

A 2 N.A. Menger PM-space (X, F, Δ) is said to be of type $(C)_g$, if there exists a $g \in \Omega$ such that

$$g\big(F(x,y,z;t)\big) \le g\big(F(x,y,a;t)\big) + g\big(F(x,a,z;t)\big) + g\big(F(a,y,z;t)\big)$$

$$\text{for each } x,y,z \in X, t \ge 0, \text{where } \Omega = \begin{pmatrix} g/g:[0,1]\to[0,\infty) \text{ is continuous,} \\ \text{strictly decreasing} \\ \text{and } g(1)=0 \text{ and } g(0)<\infty \end{pmatrix}$$

$$g(F(x,y,z;t)) \le g(F(x,y,a;t)) + g(F(x,a,z;t)) + g(F(a,y,z;t)) \text{ for each } x,y,z \in X, t \ge 0$$

where $\Omega = \{g/g:[0,1]\to[0,\infty)$ is continuous, strictly decreasing and

$$g(1) = 0 \text{ and } g(0) < \infty\}.$$

Definition 2.6:

A 2 N.A. Menger PM-space (X, F, Δ) is said to be of type $(D)_g$, if there exists a $g \in \Omega$ such that

$$g(\Delta(t_1,t_2,t_3)) \le g(t_1)+g(t_2)+g(t_3)$$
$$\text{for each } t_1,t_2,t_3 \in [0,1].$$

Remark 1:

If 2 N.A. Menger PM-space is of type $(D)_g$ then (X, F, Δ) is of type $(C)_g$.

Definition 2.7:

A sequence $\{x_n\}$ in 2 N. A. Menger PM-space (X,F,Δ) converges to x *if and only if* for each $\varepsilon > 0$, $\lambda > 0$, there exists $M(\varepsilon,\lambda)$ such that

$$g(F(x_n,x,a;\varepsilon)) < g(1-\lambda) \text{ for every } n > M.$$

Definition 2.8:

A sequence $\{x_n\}$ in 2 N. A. Menger PM-space is Cauchy sequence *if and only if* for each $\varepsilon > 0, \lambda > 0,$ there exists an integer $M(\varepsilon,\lambda)$ suchthat $g(F(x_n,x_{n+p},a;\varepsilon)) < g(1-\lambda)\ \forall n, p \ge M \text{ and } p \ge 1.$

Example 1([10]):

Let $X = R$ be the set of real numbers equipped with 2-metric defined as

$$d(x,y,z) = \begin{cases} 0 & \text{if at least two of the } x,y,z \text{ are equal} \\ 2 & \text{otherwise} \end{cases}$$

Set F(x, y, z; t) = $\dfrac{t}{t+d(x,y,z)}$.

Then (X,F,Δ) is 2 N. A. Menger PM-space with Δ as continuous t-norm satisfying

$$\Delta(r,s,t) = \min(r,s,t) \text{ or } (r.s.t)$$

Example 2([4]):

Let X = R with 2-metric defined as

$$d(x,y,z) = \min[\,|x-y|,|y-z|,|z-x|\,],$$
$$\text{for all } x,y,z \in X, t > 0.$$

Define,

$$F(x,y,z;t) = \frac{t}{t+d(x,y,z)}, \text{ with } \Delta(r,s,t) = \min(r,s,t,) \text{ or } (r.s\cdot t).$$

Then (X , F , Δ) is 2 N. A. Menger PM-space.

Example 3:

Let $X = \left\{0,1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\dots..\right\}$ equipped with 2-metric 'd'

$$d(x,y,z) = \begin{Bmatrix} 1, \text{if } x,y,z \text{ are distinct and} \\ \left\{\frac{1}{n},\frac{1}{n+1}\right\} \subset \{x,y,z\}, n \in N \\ 0, \text{ otherwise.} \end{Bmatrix}$$

Set $F(x,y,z;t) = \frac{t}{t+d(x,y,z)}$

and $\Delta(r,s,t) = min(r,s,t) \text{ or } (r.s.t)$

Then (X,F,Δ) is 2 N. A. Menger PM-Space.

Now, we define property (E.A) in 2 non-Archimedean Menger PM-space (X, F, Δ) .

Definition 2.9:

Let A and B be two self mappings of a 2 N. A. Menger PM-Space (X, F, Δ). A and B are said to satisfy property (E.A) if there exists a sequence $\{x_n\}$

such that $\lim_n Ax_n = \lim_n Bx_n = x_0 \in X$.

i.e., for a sequence x_n, there exists $x_0 \in X$ such that

$$\lim_n g\left(F\left(Ax_n, x_0, a; t\right)\right)$$
$$= \lim_n g\left(F\left(Bx_n, x_0, a; t\right)\right) = 0$$

for all $a \in X$ and $t > 0$

Example 4([4])**:**

Let X = R equipped with 2-metric 'd' defined by

$$d(x,y,z) = \min[\,|x-y|,|y-z|,|z-x|\,],$$
$$\text{for all } x,y,z \in X, t > 0$$

$$\text{Set } F(x,y,z;t) = \frac{t}{t + d(x,y,z)}$$

with $\Delta(r,s,t) = \min(r,s,t,)$ or $(r.s \cdot t)$.

Then (X, F, Δ) is 2 N. A. Menger PM-space

Define the self maps A and B on X as, $Ax = \frac{x+1}{2}$, $Bx = \frac{2x+1}{3}$

Consider the sequence

$$x_n = 1 - \frac{1}{n}, \ n = 1, 2, 3, ...$$

Then,wehave

$\lim_n g\left(F\left(Ax_n, 1, a; t\right)\right) = \lim_n g\left(F\left(Bx_n, 1, a; t\right)\right)$ for every $t > 0$ and

$a \in X$. Thus A and B satisfy the property (E.A).

In the next example we show that there are some mappings which do not satisfy the property

(E.A).

Example 5:

Let $X = [0,1]$ equipped with 2-metric 'd' defined by

$$d(x,y,z) = \min[\,|x-y|,|y-z|,|z-x|\,],$$
$$\text{for all } x,y,z \in X, t > 0$$

$$\text{Set } F(x,y,z;t) = \frac{t}{t + d(x,y,z)}$$

with $\Delta(r,s,t) = \min(r,s,t,)$ or $(r.s \cdot t)$.

Then (X, F, Δ) is 2 N. A. Menger PM-space.

Define the self maps A and B on X as, $Ax = 1 - \frac{x}{2}$, $Bx = \frac{1-x}{2}$

Consider the sequence

$$x_n = 1 - \frac{1}{n}, \ n = 1, 2, 3, ...$$

Then, we have

$\lim_n g\left(F\left(Ax_n, u, a; t\right)\right) = \lim_n g\left(F\left(Bx_n, u, a; t\right)\right) = 0$

for some $u \in X$.

Therefore,

$$\lim_n g\left(F\left(Ax_n, u, a; t\right)\right) = \lim_n g\left(F\left(1 - \frac{x_n}{2}, u, a; t\right)\right)$$

$$= \lim_n g\left(F\left(x_n, 2 - 2u, a; t\right)\right) = 0$$

and

$$\lim_n g\left(F\left(Bx_n, u, a; t\right)\right) = \lim_n g\left(F\left(\frac{1 - x_n}{2}, u, a; t\right)\right)$$

$$= \lim_n g\left(F\left(x_n, 1 - 2u, a; t\right)\right) = 0$$

We conclude $x_n \to 2 - 2u$ *and* $x_n \to 1 - 2u$, which is a contradiction. Hence A and B do not satisfy the property (E.A). Using R-weak commutativity, Pant [7 , 8] proved two common fixed Point theorems for a pair of mappings.

Theorem 1([10])**:**

Let (X, d) be a complete metric space and let f, g be R-weakly commuting self mappings of X satisfying the condition;

$d(fx, fy) \le r(d(gx, gy))$ for all $x, y \in X$, where $r: R_+ \to R_+$ such that $r(t) < t$ for each $t > 0$ if $f(X) \subset g(X)$ and if either f or g is continuous , then f and g have a unique common fixed point in X.

Theorem 2([10]):

Let (X, d) be a complete metric space and let f, g be R-weakly commuting self mappings of X satisfying the condition;

Given $\in > 0$, there exist $h(\in) > 0$ such that

(i) $\in \le d(gx, gy) < \in + h \Rightarrow d(fx, fy) < \in$

(ii) $f(x) = f(y)$

whenever $g(x) = g(y)$

If $f(X) \subset g(X)$ and if either f or g is continuous, then f and g have a unique

common fixed point in X.

Remark 2: The above two theorems do not hold if we allow both the mappings f and g to be discontinuous on X or the space X is not complete. We give the following examples in support of our claim.

Example 6:

Let $X = \left\{0, 1, \frac{1}{3}, \frac{1}{3^2}, ...\right\}$ be a metric space with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define the mappings $f, g: X \to X$ by

$$f(0) = \left(\frac{1}{3^2}\right),\ f\left(\frac{1}{3^n}\right) = \left(\frac{1}{3^{n+2}}\right),$$

$$g(0) = \left(\frac{1}{3}\right),\ g\left(\frac{1}{3^n}\right) = \left(\frac{1}{3^{n+1}}\right)$$

for $n = 0, 1, 2, ...$ respectively.

Here $d(,)x$ is complete and

$$g(X) = \left\{\frac{1}{3}, \frac{1}{3^2}, ...\right\}$$

$$\supset \left\{\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, ...\right\} = f(X),$$

f and g are R-weakly commuting for $R > 0$.

Define $\gamma(t) = \frac{1}{3}t$ for $t > 0$, both f and g are discontinuous at x = 0.

Also, we see that $d(f(0), f(1)) = \left|\frac{1}{9} - \frac{1}{9}\right| = 0$

$$d\left(f(0), f\left(\frac{1}{3}\right)\right) = \left|\frac{1}{9} - \frac{1}{27}\right| = \frac{2}{27}$$

And $$= \gamma\left(d\left(g(0), d\left(g\left(\frac{1}{3}\right)\right)\right)\right)$$

$$d\left(f(0), f\left(\frac{1}{3^2}\right)\right) = \left|\frac{1}{9} - \frac{1}{81}\right| = \frac{8}{81}$$

$$= \gamma\left(d\left(g(0), d\left(g\left(\frac{1}{3^2}\right)\right)\right)\right)$$

and so on. Also for

, $x = \frac{1}{3^n}$, $y = \frac{1}{3^m}$; $n, m = 0, 1, 2, 3, ...$

we have

$$d(f(x), f(y)) = d\left(f\left(\frac{1}{3^n}\right), f\left(\frac{1}{3^m}\right)\right)$$

$$= \left|\frac{1}{3^{n+2}} - \frac{1}{3^{m+2}}\right| = \frac{1}{3}\left|\frac{1}{3^{n+1}} - \frac{1}{3^{m+1}}\right|$$

$$= \gamma\left(d\left(g\left(\frac{1}{3^n}\right), d\left(g\left(\frac{1}{3^m}\right)\right)\right)\right)$$

$$= \gamma\left(d(g(x)), d(g(y))\right).$$

Hence all the conditions of theorem 1 are satisfied except continuity of either f or g but neither f nor g have a fixed point in X.

Example 7:

Let $X=\left\{0,1,\frac{1}{3},\frac{1}{3^2},.............\right\}$ be a metric space with the usual metric

$d(x,y)=|x-y|$ for all $x,y\in X$. Define the mappings $f,g:X\to X$ by

$$f\left(\frac{1}{3^n}\right)=\frac{1}{3^{n+2}}\ ,\ g\left(\frac{1}{3^n}\right)=\frac{1}{3^{n+1}}\ \ for\ n=1,2,3,.....$$

respectively.

Here (X,d) is not complete and

$$g(X)=\left\{\frac{1}{3},\frac{1}{3^2},.....\right\}\supset\left\{\frac{1}{3},\frac{1}{3^2},\frac{1}{3^3}.....\right\}=f(X)$$

Define $\gamma(t)=\frac{1}{3}t$ for $t>0$. Here all the conditions of theorem 1 are satisfied except the completeness of X, but neither f nor g have a fixed point in X.

On the basis of above mentioned examples, theorem 1 can be improved in two ways; either imposing certain restrictions on the space X or by replacing the notion of R-weak commutativity of mappings with certain improved notion. We choose the second option, that is the generalized notion of property (E.A).

3. RESULTS AND DISCUSSION

Theorem 3:

Let S and T be point wise R-weakly commuting self mappings of a 2 N. A. Menger PM-space (X,F,Δ) satisfying the property (E.A) and

(i) $S(X)\subset T(X)$

(ii) $g\left(F\left(Sx,Sy,a;kt\right)\right)\le g\left(F\left(Tx,Ty,a;t\right)\right)\ ;\ k>0$

(iii) $g\left(F\left(Sx,S^2x,a;t\right)\right)<\phi\left[\max\left\{\begin{array}{l}g\left(F\left(Tx,TSx,a;t\right)\right),g\left(F\left(Sx,Tx,a;t\right)\right),\\ g\left(F\left(S^2x,TSx,a;t\right)\right),g\left(F\left(Sx,TSx,a;t\right)\right),g\left(F\left(Tx,S^2x,a;t\right)\right)\end{array}\right\}\right]$ where $Sx\neq S^2x$.

If the range of S or T is complete subspace of X, then S and T have a common fixed point.

Proof:

Since S and T satisfy the property (E.A), there exists a sequence x_n in X, such that

$\lim_n Sx_n=\lim_n Tx_n=u\ ;\ u\in X$. As, $u\in S(X)\ and\ S(X)\subset T(X)$,

there exists some $w\in X$ such that $u=Tw\ where\ u=\lim_n Tx_n$

If $Tw\neq Sw$,

then $g\left(F\left(Sx_n,Sw,a;kt\right)\right)\le g\left(F\left(Tx_n,Tw,a;t\right)\right)$

Taking $n\to\infty$, We get

$g\left(F\left(Tw,Sw,a;kt\right)\right)\le g\left(F\left(Tw,Tw,a;t\right)\right)$,Hence $Sw=Tw$. Since S and T are R-weakly commuting, there exists $R>0$, such that

$$g\left(F\left(STw,TSw,a;t\right)\right)\le g\left(F\left(Sw,Tw,a;\frac{t}{R}\right)\right)=0.$$

That is $STw=TSw$ and

$SSw=STw=TSw=TTw$

If $Sw=SSw$, using (iii)

$$g\left(F\left(Sw,SSw,a;t\right)\right)<\phi\left[\max\left\{\begin{array}{l}g\left(F\left(Tw,TSw,a;t\right)\right),g\left(F\left(Sw,Tw,a;t\right)\right),\\ g\left(F\left(SSw,TSw,a;t\right)\right),g\left(F\left(SSw,TSw,a;t\right)\right),\\ g\left(F\left(Sw,SSw,a;t\right)\right)\end{array}\right\}\right]=g\left(F\left(Sw,SSw,a;t\right)\right)$$

where $Sx\neq S^2x$, a contradiction.

Hence $Sw=SSw$ and $Sw=SSw=TSw=STw=TTw$

Hence Sw is a common fixed point of S and T. The case when $S(X)$ is a complete subspace of X is similar to the above case since $S(X)\subset T(X)$.

Hence we have the theorem. Now, we give an example to illustrate the above theorem.

Example 8:

Let $X = R$ and $d : X \times X \times X \to R^+$ such that

$$d(x,y,z)=\min\left[|x-y|,|y-z|,|z-x|\right].$$

Set $F(x,y,z;t) = \dfrac{t}{t + d(x,y,z)}$ with $\Delta(r,s,t) = \min(r,s,t,)$ or $(r.s\cdot t)$

Obviously (X, F, Δ) is 2 non-Archimedean Menger PM-space.

Define $A, B : X \to X$, as

$$A(x) = \begin{Bmatrix} 1, \; if \; x \le 1 \; or \; x > 5 \\ 4 \qquad 1 < x \le 5 \end{Bmatrix},$$

$$B(x) = \begin{Bmatrix} 1, & x \le 1 \\ 2, & 1 < x \le 5 \\ x-4, & x > 5 \end{Bmatrix}$$

Consider a sequence $x_n = 5 + \frac{1}{n}$ then $Ax_n, Bx_n \to 1$ and hence A and B satisfy the property (E.A). Also, A and B are R-weakly commuting as they commute at their point of coincidence. Moreover, $A(X) \subset T(X)$

Define $\phi : [0,\infty) \to [0,\infty)$ as $\phi(0) = 0$ and $\phi(s) = \sqrt{s}$, $s > 0$

Hence all the conditions of the theorem are satisfied and '1' is common fixed point of A and B.

Setting $k = 1$ in the above theorem we get the following theorem.

Theorem 4:

Let S and T be point wise R-weakly commuting self mappings of a 2 N. A. Menger PM-space (X, F, Δ) satisfying the property (E.A) and

(i) $S(X) \subset T(X)$

(ii) $g(F(Sx,Sy,a;t)) \le g(F(Tx,Ty,a;t))$

(iii) $g\left(F\left(Sx,S^2x,a;t\right)\right) <$

$$\phi\left[\max\left\{\begin{matrix} g(F(Tx,TSx,a;t)), g(F(Sx,Tx,a;t)), \\ g\left(F\left(S^2x,TSx,a;t\right)\right), g(F(Sx,TSx,a;t)), g\left(F\left(Tx,S^2x,a;t\right)\right) \end{matrix}\right\}\right] \; where \; Sx \ne S^2x.$$

If the range of S or T is complete subspace of X, then S and T have a common fixed point. Theorem 3, ibid, has been proved by using the concept of property (E.A) which has been introduced in a recent work by Aamri and Moutawakil [5]. They have shown that the property (E.A) is more general than the notion of non compatibility. It may, however, be observed that by using the concept of non compatible maps in place of the property (E.A), we can not only prove the theorem 3 above, but in addition , we are able to show also that mappings are discontinuous at their common fixed point.

Theorem 5:

Let S and T be non compatible point wise R-weakly commuting self mappings on a 2 N. A. Menger PM-space (X, F, Δ) satisfying

(i) $S(X) \subset T(X)$

(ii) $g(F(Sx,Sy,a;kt)) \le g(F(Tx,Ty,a;t))$; $k > 0$

(iii) $g\left(F\left(Sx,S^2x,a;t\right)\right) <$

$$\phi\left[\max\left\{\begin{matrix} g(F(Tx,TSx,a;t)), g(F(Sx,Tx,a;t)), \\ g\left(F\left(S^2x,TSx,a;t\right)\right), g(F(Sx,TSx,a;t)) \\ , g\left(F\left(Tx,S^2x,a;t\right)\right) \end{matrix}\right\}\right]$$

$where \; Sx \ne S^2x.$

If range of S and T is complete subspace of X, then S and T have a common fixed point and the fixed point is the point of discontinuity.

Proof:

Since S and T are non compatible, there exists a sequence x_n in X, such that $\lim_n Sx_n = \lim_n Tx_n = u$; $u \in X$, but either $\lim_n g(F(STx_n, TSx_n, a;t)) \ne 0$ or the limit does not exist. As, $u \in S(x) \; and \; S(X) \subset T(X)$, there exists some $w \in X$ such that $u = Tw$ where $u = \lim_n Tx_n$.

If $T \neq$, then $g\left(F\left(Sx_n, Sw, a; kt\right)\right) \leq g\left(F\left(Tx_n, Tw, a; t\right)\right)$

Taking $n \to \infty$, we get

$g\left(F\left(Tx_n, Sw, a; kt\right)\right) \leq g\left(F\left(Tw, Tw, a; t\right)\right)$ Hence $Sw = Tw$. Since S and T are R-weakly commuting, there exists $R > 0$, such that

$$g\left(F\left(STw, TSw, a; t\right)\right) \leq g\left(F\left(Sw, Tw, a; \frac{t}{R}\right)\right) = 0.$$

That is

$STw = TSw$ and $SSw = STw = TSw = TTw$

If $Sw = SSw$, using (iii)

$$g\left(F\left(Sw, SSw, a; t\right)\right) < \phi\left[\max\left\{\begin{array}{l} g\left(F\left(Tw, TSw, a; t\right)\right), g\left(F\left(Sw, Tw, a; t\right)\right), \\ g\left(F\left(SSw, TSw, a; t\right)\right), g\left(F\left(SSw, TSw, a; t\right)\right), \\ g\left(F\left(Sw, SSw, a; t\right)\right) \end{array}\right\}\right] = g\left(F\left(Sw, SSw, a; t\right)\right)$$

where $Sx \neq S^2x$, a contradiction.

Hence $Sw = SSw$

and $Sw = SSw = TSw = STw = TTw$.

Hence Sw is a common fixed point of S and T. The case when $S(X)$ is a complete subspace

of X is similar to the above case since $S(X) \subset T(X)$.

We now show that S and T are discontinuous at the common fixed point $u = Sw = Tw$.

If possible, suppose S is continuous, then $\lim_n STx_n = Su = u$. R-weak commutativity implies that

$$g\left(F\left(STx_n, TSx_n, a; t\right)\right) \leq g\left(F\left(Sx_n, Tx_n, a; \frac{t}{R}\right)\right).$$

Taking $n \to \infty$, we get $\lim_n TSx_n = Su = u$ and so

$$\lim_n g\left(F\left(STx_n, TSx_n, a; t\right)\right) = 0$$

which contradicts the fact that $\lim_n g\left(F\left(STx_n, TSx_n, a; t\right)\right)$ is either non zero or nonexistent for the sequence x_n. Hence S is discontinuous.

Next, suppose that T is continuous, then $\lim_n TSx_n = Tu = u$ and $\lim_n TTx_n = Tu = u$.

Now, inequality (ii) implies that implies that

$$g\left(F\left(Sx_n, STx_n, a; kt\right)\right) \leq g\left(F\left(Tx_n, TTx_n, a; t\right)\right)$$

yields a contradiction unless $\lim_n STx_n = Tu = u$ which contradicts the fact that $\lim_n g\left(F\left(STx_n, TSx_n, a; t\right)\right)$ is either non zero or nonexistent for the sequence x_n. Thus both S and T are discontinuous.

Now, we give an example in support of theorem 5.

Example 9:

Let X = R^+ with 2-metric defined as

$$d(x, y, z) = \min[\,|x-y|, |y-z|, |z-x|\,], \quad \text{for all } x, y, z \in X, t > 0.$$

$$F(x, y, z; t) = \frac{t}{t + d(x, y, z)},$$

Define, with $\Delta(r, s, t) = \min(r, s, t,)$ or $(r.s.t)$.

Then (X, F, Δ) is 2 non-Archimedean Menger PM-space.

Define $A, B: X \to X$,

$$A(x) = \left\{\begin{array}{ll} \frac{1}{4}, & if\ 0 < x \leq \frac{1}{3} \\ \frac{1}{3}, & x > \frac{1}{3} \end{array}\right\}, \quad B(x) = \left\{\begin{array}{ll} x, & if\ x \leq \frac{1}{2} \\ \frac{1}{3}, & x > \frac{1}{2} \end{array}\right\}.$$

Then $A(X) \subset B(X)$, A and B are discontinuous.

Consider a sequence x_n defined as $x_n = \frac{1}{2} + \frac{1}{3n}$. Then

$$Ax_n, Bx_n \to \frac{1}{3}$$

but $ABx_n \to \frac{1}{4}, \; BAx_n \to \frac{1}{3}$. That is A and B are non compatible maps.

Define $\phi : [0,\infty) \to [0,\infty)$ as $\phi(0)=0$ and $\phi(s)=\sqrt{s}$, $s > 0$

Then all the conditions of above theorem are satisfied and $\frac{1}{4}$ is the common fixed point of A and B.

Theorem 6:

Let A, B, S and T be self mappings of a 2 N. A. Menegr PM-space (X, F, Δ) such that

(i)

$$g(F(Ax,By,a;t)) < \phi\left[\max\left\{\begin{array}{l} g(F(Sx,Ax,a;kt)), g(F(Ty,By,a;kt)), \\ g(F(Sx,By,a;kt)), g(F(Sx,Ty,a;kt)), \\ g(F(Ty,Ax,a;kt)) \end{array}\right\}\right]$$

(ii) (A,S) and (B,T) are R-weakly commuting

(iii)(A,S) (B,T) satisfy the property (E.A)

(iv) $A(X) \subset T(X)$ and $B(X) \subset S(X)$

If any of the range of A , B , S and T is complete subspace of X, then A , B , S and T have a

unique common fixed point.

Proof:

Suppose that (B,T) satisfy the property (E.A). Then there exists a sequence x_n in X such that $\lim_n Bx_n = \lim_n Tx_n = p$; $p \in X$. Since B(X) ⊂ S(X), there exists a sequence $y_n \in X$ such that $Bx_n = Sy_n = p$ which implies that $\lim_n Sy_n = p$.

We shall show that $\lim_n Ay_n = p$.

Using (i), we get

$$g(F(Ay_n,Bx_n,a;t)) < \phi\left[\max\left\{\begin{array}{l} g(F(Sy_n,Ay_n,a;kt)), g(F(Tx_n,Bx_n,a;kt)), \\ g(F(Sy_n,Bx_n,a;kt)), g(F(Sy_n,Tx_n,a;kt)), \\ g(F(Tx_n,Ay_n,a;kt)) \end{array}\right\}\right]$$

Taking $n \to \infty$, we get $\lim_n Ay_n = \lim_n Bx_n = p$. Suppose S(X) is a complete subspace of X.

Then $p = Su$ for some $u \in X$.

Thus $\lim_n Sy_n = \lim_n Bx_n = \lim_n Ay_n = \lim_n Tx_n = p = Su$.

First, we prove that $Au = Su$. Again using (i)

$$g(F(Au,Bx_n,a;t)) < \phi\left[\max\left\{\begin{array}{l} g(F(Su,Au,a;kt)), g(F(Tx_n,Bx_n,a;kt)), \\ g(F(Su,Bx_n,a;t)), g(F(Su,Tx_n,a;kt)), \\ g(F(Tx_n,Au,a;kt)) \end{array}\right\}\right]$$

Taking $n \to \infty$, we get $Au = Su$.

Since A and S are R-weakly commuting so $g(F(ASu,SAu,a;t)) \le g\left(F\left(Au,Su,a;\frac{t}{R}\right)\right) = 0$

Therefore, ASu = SAu and thus

AAu = ASu = SAu = SSu

Again, A(X) ⊂ T(X),

there exists $v \in X$ such that $Au = Tv$

We claim that $Bv = Tv$. Suppose that $Bv \ne Tv$ then using (i)

$$g(F(Au,Bv,a;t)) < \phi\left[\max\left\{\begin{array}{l} g(F(Su,Au,a;kt)), g(F(Tv,Bv,a;kt)), \\ g(F(Su,Bv,a;t)), g(F(Su,Tv,a;kt)), \\ g(F(Tu,Av,a;kt)) \end{array}\right\}\right]$$

$$< g(F(Au,Bv,a;kt))$$

which yields $Au = Bv$, implies $Tv = Bv$. Thus we have $Au = Su = Tv = Bv$

By R-weak commutativity of B and T, we have

$$g(F(BTv,TBv,a;t)) \le g\left(F\left(Bv,Tv,a;\frac{t}{R}\right)\right) = 0$$

Hence $BTv = TTv = BBv$.

Suppose that $AAu \ne Au$, then

$$g\left(F\left(Au,AAu,a;kt\right)\right)=g\left(F\left(AAu,Bv,a;kt\right)\right) <$$

$$\phi\left[\max\left\{\begin{array}{l} g\left(F\left(SAu,AAu,a;kt\right)\right), g\left(F\left(Tv,Bv,a;kt\right)\right), \\ g\left(F\left(SAu,Tv,a;t\right)\right), g\left(F\left(AAu,Tv,a;kt\right)\right), \\ g\left(F\left(SAu,Bv,a;kt\right)\right) \end{array}\right\}\right]$$

$$< g\left(F\left(Au,AAu,a;kt\right)\right)$$

Thus $AAu = Au$. Therefore, $AAu = Au = SAu$ That is Au is a common fixed point of A and S. Similarly, we prove that Bv is a common fixed point of B and T. Since $Au = Bv$, so Au is the common fixed point of A , B , S and T.

The proof is similar when T(X) is assumed to be complete subspace of X. The case in which A(X) or B(X) is complete subspace of X are similar to the cases in which T(X) or S(X) respectively is complete subspace of X, since A(X) $\subset$ T(X) and B(X) $\subset$ S(X).

Finally the uniqueness of fixed point can be proved easily by using (i) and hence the theorem.

Example 10:

Let $X = R^+$ equipped with 2-metric 'd'

$$d(x,y,z)=\left\{\begin{array}{l} 1, if\ x,y,z\ are\ distinct \\ \quad and \left\{\frac{1}{n},\frac{1}{n+1}\right\} \subset \{x,y,z\}, n\in N \\ 0, otherwise. \end{array}\right\}$$

Define,

$$F(x,y,z;t) = \frac{t}{t + d(x,y,z)}, \text{ with } \Delta(r,s,t) = \min(r,s,t,) \text{ or } (r.s.t)$$

Then (X,F,Δ) is 2 non-Archimedean Menger PM-space

$A, B, S, T : X \to X$

$by\ A(0)=0\ \ and\ \ A(x)=1\ if\ \ x>1$

$B(x)=0\ if\ x=0\ or\ x>6\ and\ B(x)=2\ if\ 0<x\le 6$

$S(0)=0\ and\ S(x)=2\ if\ x>0$

$T(0)=0\ and\ T(x)=4\ if\ 0<x\le 6$

$and\ T(x)=x-6\ if\ x>6$

Then A(X) $\subset$ T(X) and B(X) $\subset$ S(X). The pair (A, S) and (B, T) are R-weakly commuting as they commute at their coincidence points.

Let x_n be a sequence defined as $x_n = 6 + \frac{1}{n}$; $n \ge 1$.

Then $Bx_n, Tx_n \to 0$, which implies that (B, T) satisfies the property (E.A).

Define

$\phi:[0,\infty)\to[0,\infty)\ as\ \phi(0)=0\ and\ \phi(s)=\sqrt{s}\ ;s>0$.

Then condition (i) of above theorem is satisfied.

Thus all the conditions of theorem 6 are satisfied and '0' is the common fixed point of A, B,

S and T.

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